Math 208, Midterm 2 Name: ______________________

Signature: ______________________________________

Student ID #: ______________ Section #: I

- You are allowed a Ti-30x IIS Calculator and one 8.5 × 11 inch paper with handwritten notes on both sides. Other calculators, electronic devices (e.g. cell phones, laptops, etc.), notes, and books are **not** allowed.
- Some questions require you to explain answers: no explanation, no credit.
- Try to show your work on all questions: no work, no partial credit.
- You may use the back of the exam for scratch work: please submit any additional paper you use.
- Place a box around your answer to each question.
- Raise your hand if you have a question.

1	/10
2	/10
3	/10
4	/10
5	/10
T	/50

Good Luck!

(1) Determine bases for the rowspace (3pts), nullspace (3pts), and column space (3pts) of $A = \begin{pmatrix} 2 & 3 & 5 \\ 8 & 13 & -4 \end{pmatrix}$. Computing an echelon form, for example the RREF

$$\begin{pmatrix} 1 & 0 & \frac{77}{2} \\ 0 & 1 & -24 \end{pmatrix},$$

shows that the first two rows and columns contain pivot entries, and the third column is free Thus, we see that

$$\{\begin{pmatrix} 0 \\ 1 \\ -24 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 77/2 \end{pmatrix}\} \text{ is a basis for } \text{row}(A)$$

$$\{\begin{pmatrix} 2 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 13 \end{pmatrix}\} \text{ is a basis for } \text{col}(A)$$

$$\{\begin{pmatrix} -77/2 \\ 24 \\ 1 \end{pmatrix}\} \text{ is a basis for } \text{null}(A)$$

Note that many different answers are possible—many bases exist for any given subspace—but these answers are what the standard methods taught in class would yield.

(1pt) What is the rank of A? Ans: 2, the common dimension of the row and column space.

(2) Consider the matrix $A = \begin{pmatrix} \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 3 & 1 & 1 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix}$.

(a) (5pts) Calculate $\det(A)$ Ans: 1. This is most easily seen with cofactor expansion along the second column. The 2×2 cofactor formed by the corner entries is a plane rotation by $\pi/4$ radians ccw, hence determinant 1.

(b) (5pts) Determine whether or not A^{-1} exists, and if so compute it. A nonzero determinant from pt (a) is already enough to conclude that the inverse exists. To compute the inverse, note the reductions

$$(R_i \leftarrow \sqrt{2}R_i,\ i = 1,3) \qquad (A|I) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & \sqrt{2} & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & \sqrt{2} \end{array}\right)$$

$$(R_2 \leftarrow R_2 - 3R_1) \qquad \sim \left(\begin{array}{cccccc} 1 & 0 & -1 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 4 & -3\sqrt{2} & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & \sqrt{2} \end{array}\right)$$

$$(R_3 \leftarrow R_3 - R_1) \qquad \sim \left(\begin{array}{cccccc} 1 & 0 & -1 & \sqrt{2} & 0 & 0 \\ 0 & 1 & -2 & -3\sqrt{2} & 1 & 0 \\ 0 & 0 & 2 & -\sqrt{2} & 0 & \sqrt{2} \end{array}\right)$$

$$(R_3 \leftarrow (1/2)R_3) \qquad \sim \left(\begin{array}{cccccc} 1 & 0 & -1 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 4 & -3\sqrt{2} & 1 & 0 \\ 0 & 0 & 1 & -\sqrt{2}/2 & 0 & \sqrt{2}/2 \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc} 1 & 0 & 0 & \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 & -\sqrt{2} & 1 & -2\sqrt{2} \\ 0 & 0 & 1 & -\sqrt{2}/2 & 0 & \sqrt{2}/2 \end{array}\right),$$

hence

$$A^{-1} = \begin{pmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ -\sqrt{2} & 1 & -2\sqrt{2} \\ -\sqrt{2}/2 & 0 & \sqrt{2}/2 \end{pmatrix}$$

(3) Consider the matrix

$$R = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$$

(a) (5pts) Calculate the matrix product $R^T R$. $R^T R = I$, the 3×3 identity matrix.

(b) (5pts) What is the general solution to the following 3×3 linear system?

$$R\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

In general, a square system of linear equations with an invertible coefficient matrix has a unique solution—this follows from the "unifying theorem". In this particular case, $R^{-1} = R^T$ by part a, and thus the unique solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = R^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

(4) (10pts) Let R be the matrix from the Problem 3. Find a *nonzero* vector $v \in \mathbb{R}^3$ such that $Rv = v$. We want a nonzero vector in the nullspace of the matrices

$$R - I = (1/3)\begin{pmatrix} -2 & 2 & 2 \\ 2 & -4 & 1 \\ 2 & 1 & -5 \end{pmatrix}$$
$$\sim \begin{pmatrix} -2 & 2 & 2 \\ 2 & -4 & 1 \\ 2 & 1 & -5 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

so the answer can be any nonzero multiple of

$$v = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

(Common mistakes made here were to use the wrong coefficient matrix R, or to clear out the (1/3) before subtracting I.

(5) Let $X = \{(x, y) \in \mathbb{R}^2 \mid y = |x| \text{ and } x \geq 0\}$ (here $|\bullet|$ denotes the usual absolute value of a real number $\bullet$.)

a) (5pts) Is X a subspace of $\mathbb{R}^2$? Explain. No—$(1, 1) \in X$, but $(-1, -1) \notin X$, so X is not closed under scalar multiplication.

b) (5pts) What is the smallest subspace of $\mathbb{R}^2$ containing X? Ans: span$\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$.

This is because every vector in X is a multiple of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.